ELEMENTARY STUDENTS' CONCEPTIONS OF STEEPNESS

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In this study, we interviewed Boston-area students in Grades 2 through 7 to explore their informal knowledge of slope. We are interested in both what these students know as well as what their conceptual difficulties are as they develop an understanding of steepness. Specifically, this study investigates the question, "Which dimensions do students attend to and neglect when describing steepness?" We found that students are able to identify the steeper of two ramps or lines quite accurately; however, they have difficulties accurately describing how different dimensions of the incline contribute to steepness. These results inform teachers and curriculum developers of preconceptions and conceptual difficulties students have before taking algebra in middle school.

INTRODUCTION AND REVIEW OF RESEARCH LITERATURE

Understanding mathematics can lead to personal and professional success. The National Council of Teachers of Mathematics (NCTM, 2000) advocates that all students should study algebra. Since 2000, many states have aligned their graduation requirements with NCTM guidelines, creating a national expectation for students to pass a test covering material learned in an Algebra 1 course ("No Child Left Behind Act of 2001", 2002).

Algebra is a gatekeeper for academic success, and the algebraic topic of linearity is a gatekeeper for other algebraic concepts such as quadratic and exponential relationships (Yerushalmy, 1997). Algebra 1 students in the US, Israel, and Korea performed most poorly on linearity test questions asking for slope of a line on the coordinate plane (Greenes, Chang, & Ben-Chaim, 2007). Other studies found these conceptual difficulties: steepness and height are different, steepness is constant along an incline, slope is a ratio of differences (Cates, 2001; Lobato & Siebert, 2002).

Children who have had the opportunity to experiment with steepness may understand it to a much better extent than we see in schools. Therefore, we believe we can better prepare young children for the study of slope in middle school. The complexity of attainable cognitive tasks develops with age (Frye & Zelazo, 1998). Relational complexity (RC) theory defines complexity as the *arity* of the relation— the number of independent dimensions or variables represented concurrently. Unary relations are defined by a single attribute. If steepness is a single variable, then identifying the steeper ramp is a binary relationship. A more formal understanding of slope involves three variables: horizontal distance, vertical distance and the numerical value of slope. ternary task, attainable at a median age of 5 years (Halford, Wilson, & Phillips, 1998a). Comparing the steepness of two ramps or lines by comparing their horizontal and vertical measures involves a quaternary relation, normally attainable at an age of 11 years (Wood, 1988).

Based on cognitive complexity theory alone, as children enter school, they should be developmentally ready to accurately work with the ternary concept of slope of a ramp or line. By the end of elementary school, they should be able to compare slopes in situations where none of the measurements are held constant. Between grades 1 and 5, students can explore situations where at least one of the dimensions (hypotenuse, vertical or horizontal distance) is constant. Experience with different contexts of steepness can increase students' abilities to understand slope (Halford, Wilson, & Phillips, 1998b).

Most children have had experiences with steepness through building ramps, sliding down slides, or riding a bike up hills. Yet, research of teaching practices in the elementary schools says that pre-existing knowledge is often ignored when slope is introduced, preventing students from making connections between slope and prior knowledge (Fuson, Kalchman, & Bransford, 2005).

The first step in incorporating the mathematics of steepness into elementary schools includes creating activities and explorations which prepare the students for their middle school study of linearity. First, we need to establish what students in different grades know about steepness without instruction.

METHODOLOGY

Semi-structured clinical interview is our primary method of data collection. We have designed the interview protocol and handouts (See http://web.mit.edu/dianasc/www) to guide the interviewer through the required setup of manipulatives and questions to ask. Since the questions build one on the other and have an internal conceptual order, the protocol is quite prescriptive. However, the interviewers were encouraged to ask further questions to elucidate students' thinking.

The interview protocol consisted of five sections: an introduction and four categories of tasks. The *Concrete*, *Imagine*, *Picture*, and *Lines* sections asked the student to identify which of the two ramps presented was steeper and why. Some of the tasks required the student to construct ramps; others also required the student to draw a picture of the ramps. The student's choice of the steeper ramp was coded as *correct* or *incorrect*. For each task, we asked students to explain how they knew that the chosen ramp was steeper in two different ways. We coded their explanations in two ways: *explanation accuracy* and *conceptual category*. The codes for explanation accuracy were *correct* or *incorrect* based on whether the student used a plausible explanation to support his or her answer. The conceptual category codes included: *vertical*, *horizontal*, *hypotenuse*, *incline*, *area/space under ramp*, *speed*, *combinations*, and *other vocabulary*. We collected information from students through several sources: oral descriptions, drawings of the ramps, physical constructions, and worksheets.

We interviewed eight students attending schools in the Boston area. All of the coded interviews showed fragile understanding of steepness and provided over 250 instances of explanations of steepness. There was no evidence of correlation between the number of explanations and students' grade level (squared Pearson correlation coefficient = 0.06). The number of explanations per task also did not correlate with the student's grade (squared Pearson correlation coefficient = 0.09). There was no evidence of significant differences between the task and explanation accuracy scoring schemes (Chi Square statistic = .85, 1 df, $p > .35$). Using these methods of triangulation, we show that we obtained equivalent data from students across grade levels and that our two coding strategies showed similar results.

ANALYSIS

We found a number of surprising results from our data. We found no evidence of correlation between grade level of the student and his or her accuracy on the tasks. Task accuracy ranged from 71 to 88% and the squared Pearson Correlation Coefficient is less than 0.01.

Accuracy of explanations ranged from 45 to 90% and was also independent of the student's grade (the squared Pearson Correlation Coefficient was 0.04). We can conclude that regardless of age within the Grade 2-7 range, students are relatively accurate in determining which ramp is steeper, but have difficulties providing accurate explanations.

Accuracy of explanations did differ drastically from one task category to another. As we discussed earlier, the *Imagine* task was the most challenging for students. This is not surprising, as the questions in this category required the students to determine what information they would need in order to be able to know which ramp was steeper.

For example in the first *Imagine* scenario, we asked students whether a 20-inch board or a 10-inch board made a steeper ramp. All but one of the students claimed the 10-inch board was steeper. The correct answer is that they would need to know at least one more measure: angle, vertical height, or horizontal distance. Our second *Imagine* scenario asked if students were able to determine which ramp was steeper: one held up by 13 videos or one held up by 12 videos. The correct answer is that they would need to know at least one more measure: angle, ramp / hypotenuse length, or horizontal distance. All of our interview subjects believed the ramp with 13 videos was steeper.

Our list of the conceptual categories of students' explanations is: Incline, Vertical, Horizontal, Hypotenuse, Combinations, Area/Space under Ramp, Speed, and Other Vocabulary.

The category *Incline* includes instances where the students used synonyms or antonyms of "steep" to explain their reasoning. Sample words are: level, flat, tilt, slant, angle, diagonal, steep, pointing up. This category was the most accurately used category, with an accuracy level of 94%.

Explanations in the *Vertical* category included references to the number of videos in the tower or its height. *Vertical* was the most frequently used category which included 40.3% (104/258) of the responses. Approximately 78% of the times when students related steepness to vertical height, they were correct.

Students used *Horizontal* distance in their explanations very infrequently (6%) and inaccurately (53% correct). The difference between the accuracy of the *Vertical* and *Horizontal* explanations was significant as shown by the Chi Square test ($\chi^2 = 4.19$, df =1, $p < 0.041$), showing that students naturally form a more accurate understanding of how the vertical distance affects slope than how the horizontal distance affects it.

Some students focused on the length of the ramp, categorized as *Hypotenuse*. The two boards were the same length, but we created different hypotenuse lengths by sliding the board in and up, creating an overhang. In mathematical drawings and graphs, lines are assumed to extend infinitely. Arrows are often drawn on the end indicating that the lines go on forever. Therefore, basing the slope on the line's length is conceptually inaccurate. In fact, the slope of a line is constant regardless of the segment length. Only 11% of student explanations used the hypotenuse, and these were only 59% accurate, showing no significant difference in accuracy from the explanations using Horizontal distance ($\chi^2= 0.11$, d.f.=1, $p>0.7$).

Combinations of categories lead to a more formal understanding of steepness, namely slope. Slope is a combination (or a ratio) of the vertical and horizontal distances between any two points on a line. Every student, except the 2nd grader, correctly used a combination of categories in at least 10% of explanations.

There are several valid combinations of categories that determine steepness; using incline by itself as an explanation is sufficient, so we analyze the correct non-incline combinations. The most frequently used correct non-incline combination was vertical and hypotenuse (32%), which could be explained by the way we constructed the ramps: the only physical objects in our set-up were the board and the tapes potentially emphasizing the hypotenuse and vertical measures, respectively. It is possible that our manipulatives de-emphasized the horizontal measure.

21% of the correct combinations described by the students included the vertical and horizontal distances. Although none of the students used them in a ratio, this was the closest that they came to formalizing their conceptions of steepness into the idea of slope. Only 7% of the combinations were between the Horizontal and Hypotenuse measures. Explanations including combinations of measurements were 77% accurate, which is not significantly different from the accuracy of the Vertical explanations ($\chi^2=0.01$, df=1, $p>0.90$).

Many of the examples of combinations being used incorrectly happened in the *Imagine* part of our interview. In these questions, the students were given insufficient information and were asked to determine which of the ramps was steeper. The student would need to understand what pieces of information were missing and use them to argue their conclusion.

Four of the eight students used the *Area underneath the ramp* to explain at least one of the scenarios, with explanation accuracy 45%. The area underneath two lines is not a determining factor of steepness or slope, but in our scenarios with finite board lengths, such an explanation could be used correctly. The danger is that this reasoning cannot be extended to more general situations.

An object's speed depends also on the time it spends accelerating down the ramp. Therefore the object's final speed depends on the steepness and the length of the ramp. Using *Speed* alone to justify steepness of a ramp is incomplete, and this misconception is problematic for infinitely long lines. Only 33% of the responses coded under "Speed" were correct; this confirms our idea that the use of speed in relation to steepness can confuse students.

We tried to limit the amount of responses we coded in the *Other Vocabulary* category. One example is: "if someone were to be driving a car over it or skating over it … they would actually like land right here on the tape." This response does not fit under any other category.

Our data show that there are a number of dimensions that elementary and middle school students use to justify their reasoning about slope. The paths of reasoning are displayed in the concept map below.

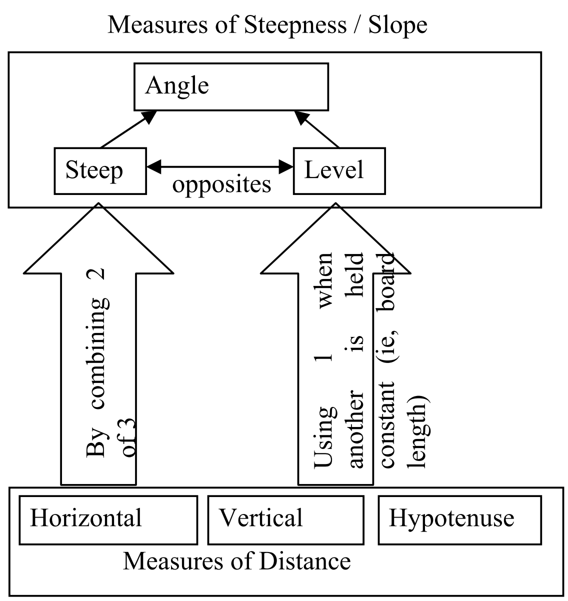

Figure 1. Concept map of the Relationships between Conceptual Categories.

Starting with the top of the concept map, we see the central mathematical idea in our research, steepness or slope. The steepness of a line is a holistic measure of the incline of the line. Slope is a mathematically defined measure of steepness: the ratio of differences of the y-coordinates (Δy) and x-coordinates (Δx) of two points on the line. The angle that a steeper line forms with the horizontal measures closer to 90 degrees, and has a slope value closer to 1. A line that is less steep will be 'flatter'; its angle with the horizontal will be closer to 0 degrees and its slope will be closer to 0. Students often supported their answers by discussing a bigger angle, more tilt, more diagonal up, etc. They were not looking at the numerical value of slope, but they were relying on their intuitive ideas of steepness.

The bottom of the concept map shows the measures of distance that can be involved in the calculation of slope or angle: horizontal, vertical, and hypotenuse distances. At least two of the three of these variables must be given in order to make mathematical conclusions about steepness. If one of the variables is held constant between two scenarios, only one other variable is needed in order to draw conclusions about the relative steepness of the two ramps. For example, if the hypotenuse is held constant (as in our questions using two boards of the same length) then the height of two ramps alone determines which of the ramps is steeper.

CONCLUSION AND FUTURE RESEARCH

The results from this study address the dimensions that students attend to and neglect when describing steepness. We showed that students most frequently refer to the vertical height of the ramp when explaining their conclusions about steepness. They also use the incline of the ramp in their justifications, as well as the hypotenuse length. To a lesser extent, they use the horizontal distance, as well as the predicted speed with which an object would roll down the ramp. Another explanation of interest is the concept of area under the ramp as an indicator of steepness. In addition, students also naturally combine some of these dimensions. Some of these combinations are redundant, while others can be used as basis for defining the mathematical concept of slope as a ratio.

All of the children had a strong intuitive understanding of steepness in familiar contexts and fragile understanding in less familiar contexts. According to RC theory, all of the students should have been capable of working with the ternary tasks that we presented them in this interview. It is possible that the students who did not successfully identify the steeper ramps in the *Concrete* section had less familiarity with the ramps in general. The only tasks that could be classified as quaternary were the *Imagine* questions where none of the dimensions were held constant. It is not surprising that students had much more difficulties completing the *Imagine* tasks.

Even when the students were able to correctly identify the steeper ramp, many used only one dimension (ie, Vertical) to describe its steepness, instead of using a combination of two features (ie, Vertical and Horizontal). Identifying two features to determine steepness is a much more complex cognitive task. When students identified a correct combination of two features, 40% of the time they used the angle of the ramp as one of the features, which is redundant.

This study had weaknesses based on our physical setup of the scenarios. Vertical height was created using a three-dimensional stack of videos and the hypotenuse was represented by the board. None of the students used grid marks on the interviewing table to describe horizontal distance. In the future, we could use a product which has equally salient horizontal and vertical dimensions.

A stronger connection needs to be made between students' experiences with ramps, understanding of the components that define steepness, and their understanding of slope. Our goal is to prepare students for the study of algebra and we must confirm a connection between our suggested experiences and their success. This study has generated many more questions than it answered. However, it has been of tremendous value to us in elucidating some of the preconceptions and misconceptions that the students bring to our classes.

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