Triangles Got Legs! A Classroom Exploration of the Largest Side and Smallest Angle in Triangles

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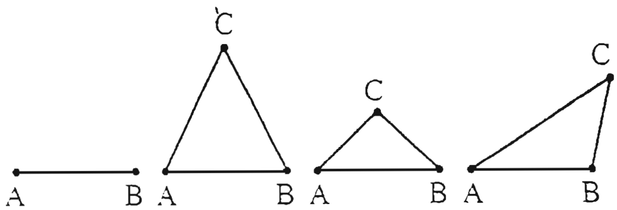
Johnny Hart's classic cartoon character, B.C., initiates a fascinating prescientific look at the secret life of clams with his discovery, "Clams got legs!" Here, too, we hope students will find rich and interesting questions to explore, beginning with the simple observation that "Triangles got legs!"—and some of those legs are longer than others!

This mathematical exploration is a small research project for young students. Problem 1 is suitable for students who have been introduced to triangles, measurement of length and comparisons of length. An extension (Problem 2) requires understanding at a conceptual level the comparison of angles of differing size. Getting the answer to these problems is not so important, rather it is the process of exploration (both the synthetic and analytic aspects) and the students' efforts to communicate and share logical thinking about spatial concepts that are important developmental exercises. Don't rush this process! Allow students to develop and test conjectures and to engage one another in acquiring this insight. If you can resist, don't read (and certainly don't teach!) the solution. Allow yourself to experience this discovery process as well—become a student! Several related activities at a variety of difficulty levels are included in order to provide still further options for the use of these ideas in the classroom.

Problem 1 (The Longest Side)

Draw a line segment AB that is to form one side of a triangle ABC. Naturally, for some choices of a third point C, the side AB will be the *longest* side of the triangle ABC, and for others it will not. The diagram below shows some of the possible choices for C.



For which points is AB the *longest* side of the triangle ABC?

Choose some other points to get a better feel for which points of C make AB the longest side. Try to identify those points C that work by shading *all* the points in the plane that made AB the *longest* side of the triangle ABC. Describe in words where these points are. Can you explain in words why these are all of the points that work?

Suggestions for Classroom Exploration

Students can be encouraged to work in small groups but should also be allowed to work alone if they prefer. Try having students record both their descriptions and the reasons for their conclusions. After everyone has made some progress, attempt to resolve the problem completely through an open class discussion. Have some students read aloud exactly what they (or other students) have written. Follow this by a class discussion of their interpretations of particular written solutions until the class achieves a common understanding of each proposed solution.

For younger students, this problem is a little more challenging than it may at first appear to be. Certainly everyone can do something (at least pick a few points and try to guess what happens for others). But in order to get an inkling of the complete solution, students will need to experiment, constantly revising their earlier guesses. Achieving a proper written description for the correct shaded region will require not only sound internalization of the concept of distance and the properties of a circle, but patience as well in precisely conveying ideas to others. It is important to note that all partial solutions are a useful basis for discussions. Discussion, for example, of what to do if "two or more sides are equal" will provide a useful exercise in establishing common conventions. Which side is the longest in such a case? Did we choose the same convention as your own class did in our solution to Problem 1 below? Does it matter?

Exploring Language and Communication

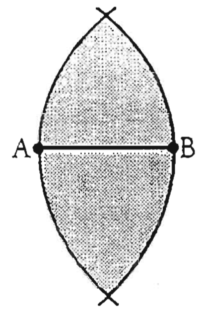
Actually writing down a description of the points that work and justifying the nature of one's answer are themselves challenging exercises, even for adult thinkers. Language can be very difficult to use properly, so that when read and interpreted by others, our meanings remain clear and unchanged. Others do not always get the meaning we intend to convey when ideas are expressed in spoken language (or even in writing). Indeed, we often clarify and develop our own understanding through our efforts to communicate ideas to others through language. This sometimes makes the writing process frustrating, but also rewarding! Some interesting thoughts bringing forth meaning through the process of communication can be found in the postmodernist writings of Maturana (1992).

For very young students, the mathematical exploration above might be creatively combined with a nonmathematical communication game. In the game of "Gossip," a story is told (whispered) to one person who tells (whispers) it to another and so on until the result comes back to the originator . . . usually in a substantially changed form, often with a hilariously different meaning! In a geometric variation of this communication game, two players (or teams) communicate ideas about shape and spatial orientation using only words. Both sides are given identical sets of geometric shapes of various colors. One side arranges their pieces inside a shallow rectangular box. Neither side may see the other's configuration. The second player or team tries to duplicate the other's arrangement, based on a series of yes or no questions and responses. A popular version of this game can be found at the San Francisco Exploratorium, where the players communicate by telephones from opposite sides of a barrier between two desks, each playing with an elegant set of a dozen colored wooden block shapes. It is surprising how difficult it is to obtain precise information without visual clues. In this game, a picture really *is* worth a thousand words! Students' first steps exploring the problems suggested here will probably involve pictures. Precise use of language in describing those pictures is a first step toward achieving critical self-awareness of the way we each express ideas and good mathematical communication.

A Solution for Problem 1

The diagram below expresses a mathematical insight. The description conveys precise understanding of that diagram, and the explanation provides a reasoned argument for the correctness of this insight. The solution consists of all three of these together.

The Diagram



The Description

The points C for which AB is the longest side of triangle ABC are precisely those that lie inside (or on) the circle with centre at B, passing through A and at the same time lie inside (or on) the circle with centre at A, passing through B.

The Explanation

- If C lies outside one of the circles, then C is farther away from A than A is from B, or C is farther away from B than B is from A. Thus, AB is not the longest side.
- If C lies inside both of the circles, then C is closer to A than A is to B, and C is closer to B than B is to A. Both other sides are therefore shorter than AB.
- If C lies on the boundary of the shaded region (the intersection of the two circles), then triangle ABC has at least two equal sides of length AB. If we allow a triangle to have more than one longest side (as in an equilateral triangle), then AB is one of the (two or three) longest sides.

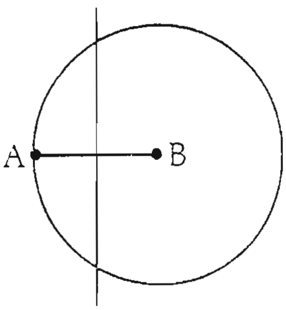
Problem 2 (The Smallest Angle)

Choose a line segment AB that is to form one side of a triangle ABC as in Problem 1. For some choices of the third point C, the interior angle at A is the smallest interior angle. As in the first exercise, try several points C to get a feel for which choices make the angle at A the smallest interior angle of the triangle ABC. Can you now indicate (by shading) all the points C for which the angle at A is the smallest interior angle of triangle ABC? Describe exactly which points these are. Explain why these are exactly the points which work. Finally, write down your description of the shaded region and your reasons why these points are exactly those that are required.

Suggestions for Classroom Exploration

Divide students into small groups of four or five to work on the problem. Have each group write down its results. Let one student from the group read aloud the results of the findings to the class (in this way no individual student need feel entirely responsible for the group's contribution). Let the class discuss what this proposed solution means to them. Together, refine both this description and the explanation until everyone is satisfied that the words clearly mean the same thing to everyone. Discuss whether or not this really resolves the problem altogether. A complete and satisfying solution (like that for Problem 1) is well within reach of a persistent class willing to experiment, conjecture and verify conclusions. Generally one can expect each small group to contribute something different toward the solution.

Teaser (Partial) Solution to Problem 2



The diagram shading, as well as the description and explanation, are left as exercises for the teacher and the class!

Further Explorations (With Increasing Levels of Abstraction)

1. Vary problems 1 and 2 by replacing largest by smallest and vice versa. Suitable for Grades 5–6 and higher.
2. Investigate the relationship between the smallest angle and largest side in triangles. Where are they located relative to one another? Do the same for the largest angle and largest side, as well as the largest angle and smallest side. Are they strictly adjacent? Opposite? Sometimes? Always? Never? Suitable for Grades 7–8 and higher.
3. At a more advanced level, investigate the relationships mentioned above by using the Law of Sines. Suitable for Grades 11–12.
4. An extension for inquiring mathematical minds: What is the situation for quadrilaterals (investigate angle/side relationships using diagrams similar to those in problems 1 and 2). Is the smallest side always adjacent to the largest angle in a triangle? In a quadrilateral? In convex polygons of four or more sides?

For a discussion of the use of "Great Explorations" in the classroom, see Friesen and Stone (1996).

References

Friesen, S., and M. G. Stone. "Great Explorations." In *Applying Research to the Classroom*, edited by E. L. Donaldson 14, no. 2 (1996): 6–11.

Maturana, H. R., and F. J. Varela. *The Tree of Knowledge: The Biological Roots of Human Understanding*. Rev. ed. Boston, Mass.: Shambala, 1992.

A store prices an item in dollars and cents so that when 4 percent sales tax is added, no rounding is necessary because the result is exactly n dollars, where n is a positive integer. What is the smallest value of n?